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Dispersion and Gibbs Phenomenon Associated with Difference Approximations to Initial Boundary-Value Problems for Hyperbolic Equations*

RAYMOND C. Y. CHIN

Lawrence Livermore Laboratory, University of California, Livermore, California 94550

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In this paper, we analyze the problem of the semidiscretized approximation for the initial boundary-value problem of the wave equation. Point-wise convergence properties for the propagation of discontinuities are investigated via a uniformly valid asymptotic expansion. An approximate error analysis using matched asymptotic expansions is constructed and compared with the asymptotic expansion of the exact solution.

INTRODUCTION

Dispersion and Gibbs phenomenon are inherent in difference approximations of hyperbolic partial-differential equations with discontinuous solutions. Their effects, smearing of the discontinuity and subsequent generation of oscillations behind the discontinuity, have been studied by a number of investigators [1–4]. All these studies are concerned with Cauchy problems and with the asymptotic behavior of the solution in the difference approximation. Consequently, they yield more precise estimates than those obtained by the use of functional analysis [5, 6].

For initial boundary-value problems, Orszag and Jayne [7] postulated that it is possible to analyze local error by replacing actual boundary and initial conditions by simpler conditions that reproduce the proper discontinuity. Their analysis of a semidiscretized approximation was shown by Chin [8], using matched asymptotic expansion techniques to account for the first truncation-error term in a Taylor series expansion of the central difference quotient.

In this paper, we analyze the problem of the semidiscretized approximation for the initial boundary-value problem of the wave equation,

$$u_t = c\sigma_x, \qquad \sigma_t = cu_x.$$

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Here, the spatial derivatives are approximated by a formally second-order-accurate central difference quotient in a stagger grid scheme. The resulting system of linear ordinary differential equations is solved exactly by Laplace transform techniques. A fundamental semidiscretized wave solution is found and utilized in the study of propagation of discontinuities. Pointwise convergence properties are investigated via a uniformly valid asymptotic expansion. An approximate error analysis using matched asymptotic expansion will be constructed and compared to the asymptotic expansion of the exact solution.

The principal results are:

(1) The sequence of approximations of the semi-discretized analog converges in the sense of Gibbs phenomenon as $\Delta x \to 0$ to the solution of the wave equation for a propagating discontinuity;

(2) The dispersive behavior giving rise to the Gibbs phenomenon is due *essentially* to the first truncation-error term in a Taylor series expansion of the difference quotient; and

(3) The higher-order terms in the Taylor expansion affect only the phase of the error waves. The relative phase error due to the higher-order terms is at most 4%.

Problem Definition

Consider the following wave propagation problem (WPP):

$$\frac{\partial u}{\partial t} = c \frac{\partial \sigma}{\partial x} \tag{1a}$$

and

$$\frac{\partial \sigma}{\partial t} = c \frac{\partial u}{\partial x}, \tag{1b}$$

for

$$x, t \in D = (x, t \mid 0 < x < L, t > 0),$$

with boundary conditions,

$$u(0, t) = f(t), \qquad u(L, t) = g(t), \qquad t > 0,$$

and homogeneous initial conditions.

We summarize the solution of the above WPP in the following representation theorem.

THEOREM 1. *The solution of the* WPP *is given by*

$$u(x, t) = \sum_{n=0}^{\infty} \left\{ f^* \left(t - \frac{2nL + x}{c} \right) - f^* \left[t - \frac{2(n+1)L - x}{c} \right] \right.$$
$$\left. + g^* \left[t - \frac{(2n+1)L - x}{c} \right] - g^* \left[t - \frac{(2n+1)L + x}{c} \right] \right\}, \tag{2}$$

where

$$f^*(\xi) = \begin{cases} f(\xi), & \xi \geqslant 0: \\ 0, & \xi < 0. \end{cases}$$

Remark. At any time, there are only four terms that are nonzero. The physical interpretation of each term is self-evident.

SEMIDISCRETIZED APPROXIMATION

Let the domain D be partitioned into strips such that

$$\Delta x = L/N$$

and

$$x_k = k\Delta x, \qquad k = 0, 1, 2, \ldots, N.$$

Following Courant, Friedrichs, and Lewy [9], we define the grid functions

$$u_k(t) = u(k\Delta x, t), \qquad k = 1, 2, \ldots, N-1,$$

and

$$\sigma_{k-(1/2)}(t) = \sigma[(k - \tfrac{1}{2})\,\Delta x, t], \qquad k = 1, 2, \ldots, N.$$

Next, each spatial derivative is replaced by its central difference approximation to obtain

$$\frac{du_k}{dt} = \frac{c}{\Delta x}\,(\sigma_{k+(1/2)} - \sigma_{k-(1/2)}), \qquad k = 1, 2, \ldots, N-1,$$

$$\frac{d\sigma_{1/2}}{dt} = \frac{c}{\Delta x}\,[u_1 - f(t)], \tag{3}$$

$$\frac{d\sigma_{k+(1/2)}}{dt} = \frac{c}{\Delta x}\,(u_{k+1} - u_k), \qquad k = 1, 2, \ldots, N-2,$$

and

$$\frac{d\sigma_{N-(1/2)}}{dt} = \frac{c}{\Delta x}\,[g(t) - u_{N-1}].$$

The initial conditions are

$$u_k(0) = 0, \qquad k = 1, 2,\dots, N-1,$$
$$\sigma_{k-1/2}(0) = 0, \qquad k = 1, 2,\dots, N.$$

Equations (3) may be combined to yield

$$\ddot{\mathbf{u}} = \mathbf{A}\mathbf{u} + \mathbf{b}, \tag{4}$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix},$$

$$\mathbf{A} = \left(\frac{c}{\Delta x}\right)^2 \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}_{(N-1)\times(N-1)}$$

and

$$\mathbf{b} = \begin{bmatrix} f(t) \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}.$$

Equation (4) is a system of second-order linear-ordinary differential equations. This system is fundamental to studying the dynamics of a lattice of point masses, each connected to its nearest neighbors by Hookean springs [10]. While the solution of Eq. (4) may be obtained in terms of contour integrals [11, 12], the representation of the solution does not offer a direct comparison with the solution of the WPP (Theorem 1). The present derivation uses Laplace transform techniques and a particular form of the transformed inverse matrix elements. The resultant solution is isomorphic to the solution of the WPP:

THEOREM 2. *The solution of the semidiscretized analog of the* WPP *is*

$$u_k(t) = \sum_{i=0}^{\infty} \Big(\bar{f}\left\{t - \frac{2}{\omega}(2Ni + k)\right\} - \bar{f}\left\{t - \frac{2}{\omega}[2(i+1)N - k]\right\}$$
$$+ \bar{g}\left\{t - \frac{2}{\omega}[(N(2i+1) - k]\right\} - \bar{g}\left\{t - \frac{2}{\omega}[N(2i+1) + k]\right\}\Big),$$

where

$$\omega = zc/\Delta x,$$

$$\bar{f}\left\{t - \frac{2}{\omega} m\right\} = 2m \int_0^t f(t-\tau) \frac{J_{2m}(\omega\tau)}{\tau} \, d\tau$$

and $J_{2m}(y)$ is the 2m-order Bessel function of the first kind.

Remark. The fundamental wave solution here is

$$\bar{f}\left\{t - \frac{2}{\omega} m\right\} = 2m \int_0^t f(t-\tau) \frac{J_{2m}(\omega\tau)}{\tau} \, d\tau$$

and, therefore, we expect this solution to converge to the exact analytic solution

$$f(t - x_m/c)$$

in some sense as $\Delta x \to 0$.

Rate of Convergence Estimates

Consider the semidiscretized wave solution

$$v_k(t) = 2k \int_0^t f(t-\tau) \frac{J_{2k}(\omega\tau)}{\tau} \, d\tau.$$

Clearly, the rate of convergence must depend on the smoothness of the function $f(t)$ (see Hedstrom [1]). Here, smoothness is measured relative to ω. To see this, we let $\bar{t} = \omega t$ and thus,

$$v_k(\bar{t}) = 2k \int_0^{\bar{t}} f\left(\frac{t-\xi}{\omega}\right) \frac{J_{2k}(\xi)}{\xi} \, d\xi. \tag{5}$$

In this paper, only the propagation of the discontinuity is considered, i.e.,

$$f(t) = H(t),$$

the Heaviside function. The exact solution to the WPP is

$$u(x, t) = H[t - (x/c)].$$

In this case, Eq. (5) becomes

$$v_k^H(\bar{t}) = 2k \int_0^{\bar{t}} \frac{J_{2k}(\tau)}{\tau} \, d\tau \tag{6}$$

and can be expressed as a sum of Bessel functions [13],

$$v_k{}^H(\tilde{t}) = 1 - J_0(\tilde{t}) - J_{2k}(\tilde{t}) - 2 \sum_{n=1}^{k-1} J_{2n}(\tilde{t}).$$

The properties of the solution Eq. (6) will be stated in the following series of lemmas.

LEMMA 1. *The $v_k{}^H(\tilde{t})$ is an oscillatory function. The local maxima and minima correspond, respectively, to the odd and even zeros of $J_{2k}(t)$.*

LEMMA 2. *For a fixed value of $\tilde{t}/2k$ as $k \to \infty$, $v_k{}^H(\tilde{t})$ has the following uniformly valid asymptotic expansion,*

$$v_k{}^H(\tilde{t}) = \tfrac{1}{3} + \int_0^{\zeta_0} A_i(-x)\, dx + O[(2k)^{-2/3}], \tag{7}$$

where

$$\zeta_0 = (2k)^{2/3} \{\tfrac{3}{2}[(\beta^2 - 1)^{1/2} - \tan^{-1} (\beta^2 - 1)^{1/2}]\}^{2/3},$$

for

$$\beta = ct/x_k = \tilde{t}/2k \geqslant 1,$$

and

$$\zeta_0 = -(2k)^{2/3} \left\{\frac{3}{2} \left[\ln \frac{1 + (1 - \beta^2)^{1/2}}{\beta} - (1 - \beta^2)^{1/2}\right]\right\}^{2/3},$$

for $\beta < 1$.

Proof. The detailed construction of the uniformly valid asymptotic expansion will be given in the Appendix. A brief sketch of the constructive method will be discussed here. The essential point is the use of the integral representation of Bessel functions of integer order,

$$J_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(t \sin x - nx)}\, dx,$$

so that steepest descent methods can be utilized. After some algebraic manipulations and interchanging integrations, we obtain

$$v_k{}^H(\tilde{t}) = 1 + \frac{1}{2\pi i} \int_{-\pi/2}^{\pi/2} e^{i(\tilde{t} \sin \xi - 2k\xi)} \cot \xi\, d\xi$$

$$- \frac{1}{2\pi i} \int_{-\pi/2}^{\pi/2} e^{i(\tilde{t} \sin \xi + 2k\xi)} \cot \xi\, d\xi.$$

A straightforward application of Laplace's method to the second integral gives

$$\frac{1}{2\pi i}\int_{-\pi/2}^{\pi/2} e^{i(\tilde{t}\sin\xi + 2k\xi)}\cot\xi\,dz = \frac{1}{2} + O[(2k)^{-1}].$$

For the first integral, the saddle points are symmetrically displaced, i.e.,

$$\xi_s = \pm\cos^{-1}(2k/\tilde{t})$$

and, therefore, as $2k/\tilde{t} \to 1$, the two saddle points coalesce. Moreover, the point of confluence is also a pole. To obtain a uniformly valid asymptotic expansion, the method of Chester, Friedman, and Ursell [14], which takes into account confluencing saddle points, is modified to include the contribution of the pole. This completes the construction.

The graph

$$\int_0^{\zeta_0} A_i(-x)\,dx$$

vs ζ_0 for $-8 \leqslant \zeta_0 \leqslant 10$ is given in Fig. 1.

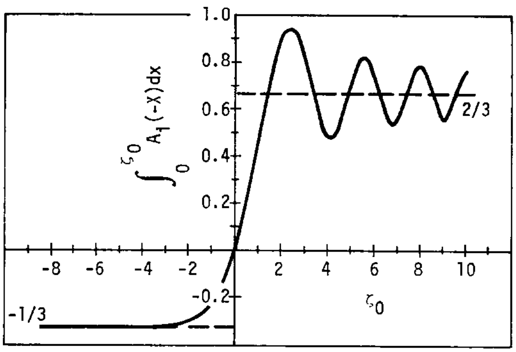

FIG. 1. The integral of the Airy function.

For $|\zeta_0| \gg 1$, the following asymptotic expansions are known [13].

$$\int_0^{\zeta_0} A_i(-x)\,dx \sim \frac{2}{3} - \frac{1}{\pi^{1/2}}\zeta_0^{-3/4}\cos\left(\frac{2}{3}\zeta_0^{3/2} + \frac{\pi}{4}\right) \tag{8a}$$

and

$$\int_0^{|\zeta_0|} A_i(x)\,dx = \frac{1}{3} - \frac{1}{2\pi^{1/2}}|\zeta_0|^{-3/4}\exp\left(-\frac{2}{3}|\zeta_0|^{3/2}\right). \tag{8b}$$

Remark. The analysis leading to (7) is valid for $k \to \infty$. From the definition of

$$k = x_k/\Delta x,$$

two interpretations are possible. For x_k fixed, $k \to \infty$ implies $\Delta x \to 0$. This case is concerned with the rate of convergence and the character of the limit function. On the other hand, for Δx fixed, $k \to \infty$ implies $x_k \to \infty$. As a byproduct, therefore, an estimate is obtained of the dispersive effect at large distances, and thus long times.

The rate of convergence estimates follows immediately from Lemma 2 and Eq. (8) and, therefore, will be stated without proof.

THEOREM 3 (Gibbs phenomenon). *Given any $\epsilon > 0$ and $\delta > 0$, as small as we please,*

(1) *there exists a Δx_1, such that*

$$| v_k^H(t; \Delta x_1) - 0.00 | < \delta$$

and

$$1 - (ct/x_k) > \epsilon;$$

(2) *there exists a Δx_2, such that*

$$| v_k^H(t; \Delta x_2) - 1.00 | < \delta$$

and

$$(ct/x_k) - 1 > \epsilon.$$

(3) *Thus, let $\Delta x = \max(\Delta x_1, \Delta x_2)$; then*

$$\delta < | v_k^H(t; \Delta x) - H[t - (x_k/c)]| \leqslant \tfrac{1}{3} + \int_0^{A_1} A_i(-x)\, dx = 1.2747$$

and

$$|(ct/x_k) - 1 | < \epsilon.$$

For the case of Δx fixed, we can estimate the dispersive effect for $x_k \to \infty$ by examining the location of the first maximum relative to the wave front. To do so, we make use of Olver's results on uniformly valid asymptotic expansions of Bessel functions of larger order [15] for estimating the location of the first zero. In terms of (x, t) variables, we have

$$ct_1 = x_k + \tfrac{1}{2}(\Delta x)^{2/3} x_k^{1/3} A_1 + (3/40)(A_1 \Delta x^{2/3})^2 x_k^{-1/3} + \cdots,$$

where $A_1 = 2.3381$ is the first zero of the Airy function $\mathrm{Ai}(-y)$. This clearly shows that the first maximum spreads as $x_k^{1/3}$ away from the wave front.

Error Analysis by the Method of Matched Asymptotic Expansions

In this section, we shall show that the method of matched asymptotic expansions is a viable technique for obtaining maximum norm-error estimates. The error wave equation analyzed is obtained by truncating terms higher than Δx^2 in a Taylor series expansion of the difference quotient, and then solved by the method of matched asymptotic expansions. The solution is compared with the asymptotic analysis in the previous section. This comparison yields the relative importance of the higher order terms in the Taylor expansion of the difference quotient. The result shows that the higher-order terms contribute only toward the phase of the error wave far from the wave front, and a maximum relative error of 4% in the phase is obtained.

The problem to be analyzed is the propagation of a step discontinuity considered in the last section, i.e.,

$$u(x, t) = H[t - (x/c)].$$

Following Chin [8], we expand $u(x_k \pm \Delta x, t)$ of Eq. (4) in a Taylor series about (x_k, t) and neglect terms with orders higher than Δx^2 to obtain

$$\frac{\partial^2 w}{\partial t^2} = c^2 \left\{ \frac{\partial^2 w}{\partial x^2} + \frac{(\Delta x)^2}{12} \frac{\partial^4 w}{\partial x^4} \right\}. \tag{9}$$

Here, $w(x, t)$ denotes the approximate solution. Equation (9) will then be solved by the method of matched asymptotic expansions.

The detailed construction of the matched asymptotic expansion parallels that of Chin [8] and, therefore, will be omitted. However, a brief description will be given. Essentially, an outer solution valid outside the neighborhood of the wave front $(x = t)$ is matched to an inner solution valid in the neighborhood of the wave front. The inner expansion is the solution of the equation

$$\frac{\partial^2 w^I}{\partial \xi\, \partial t} = \frac{1}{24} \frac{\partial^4 w^I}{\partial \xi^4},$$

where the stretch variable is

$$\xi = (c/\Delta x)^{2/3} [t - (x/c)].$$

The resultant matched asymptotic expansion is

$$w(x, t\colon \Delta x) = \tfrac{1}{3} + \int_0^{2\xi/t^{1/3}} A_i(-\tau)\, d\tau + O[(\Delta x/c)^{2/3}]. \tag{10}$$

Comparisons of Equations (10) *and* (7)

The difference between (10) and (7) is in the limit of integration, or the phase of the error wave,

$$\zeta_A = 2\xi/t^{1/3} = 2(x/\Delta x)^{2/3} (ct/x)^{-1/3} [(ct/x) - 1], \tag{11a}$$

as compared to

$$\zeta = \begin{cases} (2k)^{2/3} \left\{\frac{3}{2} [(\beta^2 - 1)^{1/2} - \tan^{-1} (\beta^2 - 1)^{1/2}]^{2/3}\right\}, & \beta \geqslant 1, \\ -(2k)^{2/3} \left\{\frac{3}{2} \left[\ln \frac{1 + (1 - \beta^2)^{1/2}}{\beta} - (1 - \beta^2)^{1/2}\right]\right\}^{2/3}, & \beta < 1. \end{cases} \tag{11b}$$

Noting that $ct/x = \beta$ and $x_k = k\Delta x$, we have the ratio of the two integration limits given by

$$\frac{\zeta}{\zeta_A} = \frac{3^{2/3}}{2} \frac{\beta^{1/3}}{\beta - 1} [(\beta^2 - 1)^{1/2} - \tan^{-1}(\beta^2 - 1)^{1/2}]^{2/3}, \qquad \beta \geqslant 1.$$

Examining the behavior of this ratio, we find that

$$\frac{\zeta}{\zeta_A} = \begin{cases} 1 + (1/30)(\beta - 1) + O[(\beta - 1)^2], & \beta - 1 \ll 1, \\ (3^{2/3}/2)\{1 - [(\pi - 3)/3](1/\beta) + O(1/\beta^2)\}, & \beta \gg 1. \end{cases}$$

Moreover, ζ/ζ_A is a monotonically increasing function of β; therefore, we conclude that

$$1 \leqslant \zeta/\zeta_A \leqslant 3^{2/3}/2 = 1.040042, \tag{12}$$

for $1 \leqslant \beta < \infty$. Similar results hold for $0 < \beta \leqslant 1$. The ratio of the phases, ζ/ζ_A is shown in Fig. 2.

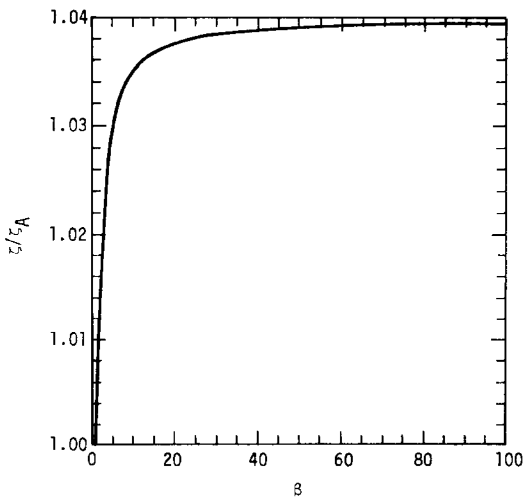

FIG. 2. Ratio of the phases.

Conclusions

The results, namely a comparison of the approximate solution (10) and the asymptotic expansion of the exact solution (7) for $\Delta x \to 0$ in the case of a propagating step discontinuity, clearly demonstrate that: (1) the dispersive character of the semidiscretized solution is primarily due to the first-truncation-error term, i.e., $(\Delta x)^2/12\ \partial^4 w/\partial x^4$, and (2) the method of matched asymptotic expansions is a viable technique for analyzing local truncation error.

The application of the method of matched asymptotic expansions for problems with variable coefficients is more detailed, but poses no essential additional problem. The treatment of nonlinear problems would appear to involve nontrivial extensions of the technique. In either case, in view of the sharpness of (12), the applicability of the method to local truncation-error analysis used in this section seems to hold great promise.

Appendix

In this appendix, we develop a uniformly valid asymptotic expansion for the integral

$$I = \frac{1}{2\pi i} \int_{-\pi/2}^{\pi/2} e^{i(t \sin \zeta - 2k\zeta)} \cot \zeta \, d\zeta \tag{A.1}$$

for $k \to \infty$. The method of Chester, Friedman, and Ursell is modified in light of Bleistein's procedure for finding uniform asymptotic expansion with the stationary point near algebraic singularity [16].

Analysis.

The transformation

$$\sin \zeta = \eta, \qquad -1 \leqslant \eta \leqslant 1,$$

substituted into (A.1) gives

$$I = \frac{1}{2\pi i} \int_{-1}^{1} e^{i2k(\beta\eta - \sin^{-1}\eta)} \, d\eta/\eta, \tag{A.2}$$

where

$$\beta = t/2k.$$

The saddle points are located at

$$\eta = \pm [1 - (1/\beta^2)]^{1/2}.$$

As $\beta \to 1$, $\eta \to 0$ and the confluence of two saddle points occurs. Moreover, $\eta = 0$ is a pole of the integrand. For the sake of convenience, we give details only for the case $\beta > 1$.

Following Chester, Friedman, and Ursell, we perform the following change of variables,

$$\beta\eta - \sin^{-1}\eta = \zeta(\beta)\, u - \tfrac{1}{3}u^3. \tag{A.3}$$

Here, $\zeta(\beta)$ is determined by requiring the transformation (A.3) to be regular and one-to-one. This implies that $d\eta/du \neq 0$ or ∞, and

$$\left[\beta - \frac{1}{(1-\eta^2)^{1/2}}\right]\frac{d\eta}{du} = \zeta(\beta) - u^2.$$

Thus, we may have

$$\zeta(\beta) - u^2 = 0 \tag{A.4}$$

or

$$u = \pm\zeta^{1/2}$$

only at the saddle points,

$$\eta = \pm[1 - (1/\beta^2)]^{1/2}.$$

Therefore, we obtain

$$\zeta(\beta) = \{\tfrac{3}{2}[(\beta^2-1)^{1/2} - \tan^{-1}(\beta^2-1)^{1/2}]^{2/3}\}, \qquad \beta \geqslant 1. \tag{A.5}$$

In the neighborhood of each point in the η plane, (A.3) is locally one-to-one analytic in all veriables (see Bleistein [17]).

Substituting (A.3) into (A.2), we have

$$I = \frac{1}{2\pi i}\int_{-\bar{u}}^{\bar{u}} e^{i(2k)[\zeta u-(1/3)u^3]}\left(\frac{1}{\eta}\frac{d\eta}{du}\right) du, \tag{A.6}$$

where $\bar{u}$ is the solution of

$$\tfrac{1}{3}u^3 - \zeta u + [\beta - (\pi/2)] = 0.$$

To evaluate $(1/\eta)(d\eta/du)$, we must assume that (A.3) is one-to-one analytic where required. In contrast to Chester, Friedman, and Ursell, the method of Bleistein [17] is applied to generate the coefficients for the expansion of $(u/\eta)(d\eta/du)$. We set

$$\frac{u}{\eta}\frac{d\eta}{du} = a_0 + a_1 u + a_2 u^2 + u^2(u^2 - \zeta)\, G_1(u). \tag{A.7}$$

The coefficients a_0, a_1, a_2, and $G_1(u)$ must be determined.

Evaluating (A.7) at $u \to 0$ and $u = \pm\zeta^{1/2}$, we obtain

$$a_0 = \lim_{u\to 0}\left(\frac{u}{\eta}\frac{d\eta}{du}\right),$$

$$a_1 = \left[\left(\frac{u}{\eta}\frac{dn}{du}\right)_{u=+\zeta^{1/2}} - \left(\frac{u}{\eta}\frac{d\eta}{du}\right)_{u=-\zeta^{1/2}}\right]\Big/2\zeta^{1/2},$$

$$a_2 = \left[\left(\frac{u}{\eta}\frac{d\eta}{du}\right)_{u=+\zeta^{1/2}} + \left(\frac{u}{\eta}\frac{d\eta}{du}\right)_{u=-\zeta^{1/2}} - 2a_0\right]\Big/2\zeta.$$

It follows from L'Hospital's rule that

$$a_0 = 1, \qquad a_1 = 0, \qquad \text{and} \qquad a_2 = -\frac{1}{\zeta}\left[\frac{\sqrt{2}}{\beta}\left(\frac{\zeta}{\beta^2 - 1}\right)^{1/4} - 1\right].$$

Inserting (A.7) into (A.6), we have

$$\begin{aligned} I = {} & \frac{1}{2\pi i}\int_{-\bar{u}}^{\bar{u}} e^{i(2k)(\zeta u - 1/3u^3)}\left(\frac{1}{u} + a_2 u\right) du \\ & + \frac{1}{2\pi i}\int_{-\bar{u}}^{\bar{u}} e^{i(2k)(\zeta u - 1/3u^3)}\, u(u^2 - \zeta)\, G_1(u)\, du. \end{aligned} \tag{A.8}$$

The last integral may be rewritten after integration by parts to yield

$$\begin{aligned} & \frac{1}{2\pi(2k)}\, e^{i(2k)(\zeta u - 1/3u^3)}\, uG_1(u)\Big|_{-\bar{u}}^{\bar{u}} \\ & \qquad - \frac{1}{2\pi(2k)}\int_{-\bar{u}}^{\bar{u}} e^{i(2k)(\zeta u - 1/3u^3)}\,[G_1(u) + uG_1'(u)]\, du. \end{aligned}$$

Once again, we set

$$G_1(u) + uG_1'(u) = \tilde{a}_0 + \tilde{a}_1 u + \tilde{a}_2 u^2 + u^2(u^2 - \zeta)\, G_2(u)$$

and repeat the above procedure to generate recursively the desired coefficients.

Next, we restrict our attention to the evaluation of the first two integrals of (A.8), which may be rewritten as

$$\left\{-\frac{1}{\pi}\int_0^{\infty} + \int_{(2k)^{1/3}\bar{u}}^{\infty}\right\}(1 + u)\, du \sin\left(\frac{1}{3}u^3 - \zeta_0 u\right),$$

where

$$\zeta_0 = (2k)^{2/3}\,\zeta.$$

Using the integral representation of the Airy function and integrating by parts, we have for $\beta \geqslant 1$

$$I = \int_0^{\zeta_0} A_i(-x)\, dx - \frac{1}{6} + \frac{1}{(2k)^{2/3}}\, a_2 A_i'(-\zeta_0) + 0\left(\frac{1}{2k}\right). \tag{A.9}$$

For $\beta < 1$, the analysis is essentially the same, and we get

$$\zeta_0 = -\left\{\frac{3}{2}\left[\ln \frac{1 + (1 - \beta^2)^{1/2}}{\beta} - (1 - \beta^2)^{1/2}\right]\right\}^{2/3} (2k)^{2/3}$$

and

$$I = \int_0^{|\zeta_0|} A_i(x)\, dx - \frac{1}{6} + \frac{1}{(2k)^{2/3}}\, a_2 A_i'(|\, \zeta_0 \,|) + 0\left(\frac{1}{2k}\right).$$

In conclusion, we have

$$I = \int_0^{\zeta_0} A_i(-x)\, dx - \frac{1}{6} + 0\left(\frac{1}{2k}\right)^{2/3},$$

where

$$\zeta_0 = \begin{cases} (2k)^{2/3} \left\{\frac{3}{2}\left[(\beta^2 - 1)^{1/2} - \tan^{-1}(\beta^2 - 1)^{1/2}\right]\right\}^{2/3}, & \beta \leqslant 1, \\ -(2k)^{2/3} \left\{\frac{3}{2}\left[\ln \frac{1 + (1 - \beta^2)^{1/2}}{\beta} - (1 - \beta^2)^{1/2}\right]\right\}^{2/3}, & \beta \leqslant 1. \end{cases}$$

The necessity of changing variables for $\beta < 1$ reflects the change in the Debye contours for computing expansion of Bessel function of large order.

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